Twisted Curves whose Tangents belong to a Linear Complex.

By Virgil Snyder.

It was shown by Lie that the problem of finding the curves whose tangents belong to a non-special linear complex is co-extensive with that of the enumeration of minimum developables. By duality, the latter reduces to the problem of curves on a quadric cone, hence if these latter curves be known, the corresponding complex curves are determined by means of two transformations.

Since the properties of lying on a quadric cone and of belonging to a linear complex are purely projective, it will be desirable to replace the minimum developable by any developable on which a conic lies, and to avoid both metrical and imaginary elements of the transformation. It is the purpose of this paper to analyze the singularities of the Noether line-point transformation and of duality, then to give a classification of complex curves of orders one to six, and to discuss the forms of singularities which complex curves can have.

1. *Curves on a quadric cone.*

Let an algebraic curves pass b times through the vertex of a quadric cone K_2 and cut each generator in a other points. Its order is then $2a+b$. By projecting this curve from an arbitrary point O of K_2 through which it does not pass, and cutting the projecting cone with an arbitrary plane π we obtain a plane curve of order $2a+b$ in $(1, 1)$ correspondence with the curve on K_2. Let the generator through O cut π in O_1. It cuts the curve on K_2 in $a+b$ points, the tangents to a of which lie in the tangent plane to K_2 along OO_1. In the plane curve we have a branches touching each other, with an independent b-fold point at the point of tangency. This singularity is equivalent to

$$a(a-1)+\frac{b}{2}(b-1)+ab$$

double points. The plane curve can have no other singularities except the pro-

jections of actual nodes and cusps of the curve on K_2. The class of the plane curve is equal to the rank of the twisted curve, hence

$$r = 2a(a+b) - 2H - 3\beta$$
$$p = (a-1)(a+b-1) - H - \beta.^*$$

An arbitrary line will cut r tangents to c_{2a+b}. If the line pass through the vertex of K_2, $2a$ of these tangents will evidently lie in the tangent planes to K_2 through the line. The b tangents at the vertex will each count for two, hence there are $2a(a+b) - 2H - 3\beta - 2(a+b)$ generators of K_2 which touch the curve, the point of tangency not being at the vertex. The osculating planes at these points are stationary planes passing through the vertex. The other stationary planes do not pass through the vertex. The osculating plane at a simple cusp must pass through the vertex, as must also the osculating plane at at a point of inflexion, the latter being a plane of six point contact, being the equivalent of three stationary planes. The inflexional tangent must pass through the vertex. A cusp or linear inflexion may exist at the vertex; higher point singularities may exist at other points of the curve, without requiring the osculating plane to pass through the vertex. The effect of these singularities will be the same as that of their projections in π.

If the equation of K_2 be $y^2 - 4xz = 0$, and the coordinates of O be $(0,1,0,0)$ the relation between the point (x, y, z, w) on the cone and the point (ξ, η, ζ) in π may be represented thus:

$$\rho w = \xi\zeta, \quad \rho x = \xi^2, \quad \rho y = 2\xi\eta, \quad \rho z = \eta^2.$$

The equation of the tacnodal tangent, intersection of π and the tangent plane to K_2 along OO_1 is $\xi = 0$, the point of tangency being $(0, 0, 1)$. A plane curve of order $2a+b$ having a branches touching each other and another b-fold point at the point of contact is the projection of a conical c_{2a+b}. Every other singularity of the plane curve will project into a similar singularity of the conical curve.

2. *Duality.*

By means of the point (x, y, z, w) and plane (u, v, s, t) duality wherein $ux + vy + sz + tw = 0$, the cone $y^2 - 4xz = 0$ goes over into the conic c_2:

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* The letters r, p, H, β and others that will be used have the same meaning as in Pascal's (German) Repertorium, vol. 2, p. 228, col. 4. This formula was proved by Ed. Weyr and by Baule by means of abelian integrals. It was generalized to apply to curves on a cone of any order by Sturm, *Math. Annalen*, vol. 19, p. 487, where the preceding memoirs are also cited.

$y^2 - xz = 0$, $w = 0$, the vertex $(0, 0, 0, 1)$ into the plane $w = 0$, the generators of K_2 into the tangents of c_2. The curve c_{2a+b} becomes a developable D_r of rank r, having c_2 for a-fold curve, having b-tangents with w for torsal plane, and $2a(a+b) - 2H - 3\beta - 2(a+b)$ other tangents in $w = 0$ which are simple generators of the developable. Stationary planes go into cusps; those passing through the vertex into those on c_2; cusps go into stationary planes, the point of contact being on c_2. The various singularities of c_{2a+b} which have no special relation to the vertex of K_2 go over into dual singularities having no particular relation to $w = 0$.

3. *The Noether line-point transformation.*

The details of this transformation were given in my Göttingen dissertation,* but in metrical form, and with reference to spheres rather than minimum lines as generators.

Consider the two bi-linear equations

$$x_1 x + y_1 y + w_1 w = 0,$$
$$y_1 x + z_1 y + w_1 z = 0,$$

in which x, y, z, w and x_1, y_1, z_1, w_1 are coordinates of points in two spaces S, S_1. When (x_1) describes S_1 the corresponding line in S describes the linear complex $\psi \equiv p_{14} + p_{23} = 0$. When (x) describes S, the corresponding line in S_1 describes the quadratic complex $\Omega \equiv p_{14}p_{34} - p_{42}^2 = 0$, consisting of the lines which cut the conic $y_1^2 - x_1 z_1 = 0$, $w_1 = 0$.†

The lines of a pencil in ψ go into the points of the line in Ω which is the image of the vertex. A curve in ψ goes over into a developable in Ω. The ∞^2 lines of a linear congruence contained in ψ, defined by

$$a p_{12} + b p_{34} + c p_{31} + d p_{24} + e p_{14} + f p_{23} = 0$$

become the ∞^2 points of the quadric surface R_2,

$$b(x_1 z_1 - y_1^2) - w_1(a w_1 + c z_1 + d x_1 + (f - e) y_1) = 0,$$

which may be said to be the image of the two directrices of the congruence. These directrices are conjugate polars as to ψ, hence to any line in S corresponds

* Ueber die linearen Complexe der Lie'schen Kugelgeometrie, Göttingen, 1895. Many of the details are also given in Lie-Scheffers, Berührungstransformationen, 1897.

† This form of the equations is given by Wiman, Klassifikation af regelytorna af sjette Graden, Lund dissertation, 1892. The transformation of singularities is not discussed. See also Jessop's Treatise on the line complex, pp. 121–3.

an R_2 in S_1 containing c_2; the second system of generators of R_2 are the images of the points of the conjugate polar of the given line as to ψ. When the directrices coincide, the single directrix belongs to ψ; its points go over into the quadric cone whose vertex is the image of the given line, and which contains c_2. If the directrices of the congruence cut l, $x=0$, $y=0$, i.e., if $b=0$, R_2 breaks up into $w_1=0$ and the plane $a w_1 + c z_1 + d x_1 + (f-e) y_1 = 0$; the points of the directrices go over into the two pencils in the second plane, whose vertices are at the points in which the plane cuts c_2. A directrix belonging to ψ and cutting l goes over into a tangent plane to c_2.

The ∞^2 lines of ψ which cut l go over into the ∞^2 points of $w_1=0$. Conversely, all the points of $w_1=0$ go over into the same line l. The relation between the points in S_1 and directions from points of l is expressed by $x_1 : y_1 : z_1 = dp_{24} : dp_{23} : dp_{31}$. To a point x_1, y_1 on c_2 correspond lines of ψ in $x_1 x + y_1 y = 0$, the vertex being on l. To the ∞^2 lines in S_1 through this point correspond the ∞^2 points of the plane $x_1 x + y_1 y = 0$. To the lines of a pencil containing the tangent to c_2 and vertex on c_2 correspond ∞^1 points on a line of ψ in the plane. When a curve in S_1 has a double point on c_2, it goes over into two lines of ψ from the same point of l. When the plane of the two tangents contains the tangent to c_2 at the double point the image lines in S coincide.

A line in $w_1=0$ cuts c_2 in two points, but we need not consider such cases here, since every generator of a developable containing c_2 and lying in the plane of the latter must be tangent to c_2. If then the developable be of order r and contains c_2 as an a-fold line, the corresponding curve in S will be of order $r-a$, having $r-2a$ points on l.

Cusps and linear inflexions in S_1 go into linear inflexions and cusps in S, provided they do not lie on c_2. A cusp of D_r on c_2 goes into a point of c_{r-a} on l, and a point of stationary contact on c_2 goes over into a line of ψ cutting l which has a point of tangency not on l. A generator of D_r in w_1, and having w_1 for torsal plane goes into a point of c_{r-a} on l, having l for tangent. A double tangent plane goes into a line of ψ cutting l which is a bisecant of c_{r-a}.

On combining these two transformations we see that every curve on K_2 goes over into a curve in ψ and conversely. In this depiction, any complex whatever that is not special can by linear point transformation be reduced to this form ψ. The same complex curve may give rise to different curves on K_2 by choosing l differently. Any line of the complex to which a curve may belong can be taken as l, and any conic in S_1 may be taken for c_2.

A curve of order $2a+b$ on K_2 goes over into a complex curve of order $2a(a+b)-2H-3\beta-a$. The line l has $2a(a+b)-2H-3\beta-2a$ points upon the curve, of which $2b$ are absorbed in b points of contact, images of the b tangents to c_{2a+b} at the vertex of K_2. The other points of the complex curve on l are images of the stationary planes which pass through the vertex of K_2. There are β tangents of the complex curve which cut l, and also H bisecants belonging to ψ cut l. The genus of the complex curve is the same as that of the conical curve. To every stationary plane of the conical curve not passing through the vertex corresponds an inflexion of the complex curve, hence we have the following theorem: *At the point of contact of a stationary plane of a curve contained in a linear complex the tangent line has at least three point contact.**

If a tangent having more than two-point contact with the conical curve does not pass through the vertex, the point of contact must be a singular point, hence: *The osculating plane at a cusp of a curve contained in a linear complex must have at least five-point contact with the curve at the cusp.*

Finally, since every complex curve is self-dual in such a way that every generator of the developable formed by its tangents goes into itself when inverted in the complex† it follows that $m=n$, $\alpha=\beta$, $H=S=0$, $h=g$, $x=y$. Further, the cusps and points of stationary contact must be in united position, provided the latter is not an inflexion.‡

4. *Particular examples.*

The c_3, $a=1$, $b=1$ or $(1, 1)$ on K_2 goes over into c_3 in ψ, l being a tangent.

A nodal c_4 $(2, 0)$, vertex not at the node, is of rank 6; two tangents pass through the vertex and it has two other stationary planes. It becomes a c_4 having two linear inflexions; l is a bisecant of c_4 and lies in the osculating plane of each point of intersection, hence no other tangents can cut l.

These two curves were long since known to belong to a complex. Now consider the c_4 having its node at the vertex of K_2, type $(1, 2)$. No tangents

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*This theorem can also be proved geometrically. Another proof was given by Picard, Application de la théorie des complexes linéaires à l'étude des surfaces et des courbes gauches. *Annales de l'école normale*, 1877.

†See Wilczynski, *Math. Annalen*, vol. 58, p. 249, and Sisam, *Bulletin Amer. Math. Society*, vol. 10, p. 440.

‡The word duality is used in a much broader sense by Fiedler in the Vierteljahresschrift der Naturforsch. Gesellschaft in Zürich, vol. 20, in which he enumerates possible self-dual curves, simply according to the Cayley characteristics. Complex curves are not discussed.

pass through the vertex except the two at the vertex. We have in ψ a c_5 of which l is a bitangent and having four inflexions.*

The cuspidal c_4, vertex not at cusp goes over into c_3 having one point on l; one tangent to c_3 cuts l. If the cusp be at the vertex, the complex curve is a c_4 having l for one inflexional tangent, and one other.

An interesting curve is afforded by the quartic of the first kind (2, 2). The complex curve is of order six and genus one. The line l is a quadri-secant, the points of intersection being harmonic. The 12 inflexional tangents are arranged in three tetrads such that the two transversals of a tetrad also cut l and are conjugate polars as to ψ. The lines in the stationary planes which cut l and belong to ψ are so arranged that a line of ψ cutting any three will also cut a fourth. The conical curve c_{2n+1}:

$$x=(2n+1)\lambda^{2n+1}, \quad y=2(2n+1)\lambda^{n+1}, \quad z=(2n+1)\lambda, \quad w=2n,$$

has all points of the plane $x=0$ at the vertex of the cone, the generator $x=0$, $y=0$ being a tangent of $(n+1)$-point contact. The point (1, 0, 0, 0) is an $(n-1)$-fold point, the osculating plane $w=0$ having $(2n+1)$-point contact, the tangent line having $2n$-point contact. It goes into the complex curve:

$$x=(2n+1)\lambda^{2n+1}, \quad y=(2n+1)\lambda^{n+1}, \quad z=(2n+1)\lambda^{n}, \quad w=-1,$$

which has two n-fold points, the tangents having $(n+1)$-point contact, and osculating planes having $(2n+1)$-point contact. The points are (0, 0, 0, 1) and (1, 0, 0, 0). Thus a cusp of this nature on the complex curve and having the fundamental line for tangent goes into the vertex of the cone K_2, the tangent having $(n+1)$-point contact, and the point itself simple on the conical curve. The same singularity, but having no particular relation to the fundamental line goes into a singular point of the same order as that on the complex curve, but the tangent has maximum contact. Neither the tangent nor the osculating plane passes through the vertex of the cone. Similarly, the conical curve

$$x=n\lambda^{n}, \quad y=2n\lambda^{n-1}, \quad z=n\lambda^{n-2}, \quad w=2,$$

having an $(n-2)$-fold point at the vertex, and a stationary plane of n-point

* The rational quintic with four inflexions and a double tangent has also been found by Dr. Colpitts in his Enumeration of twisted quintic curves, Cornell dissertation, 1906, by a totally different method.

contact at (0, 0, 0, 1), the tangent having simple contact, goes into the complex curve

$$x = n\lambda^n, \quad y = n\lambda^{n-1}, \quad z = n\lambda, \quad w = 2 - n,$$

having two tangents of $(n-1)$-point contact, but no multiple points.*

As an example of a less symmetric curve, the conical c_6: $x = \lambda^6$, $y = 20\lambda^3$, $z = 100$, $w = 6\lambda$ goes into the complex c_6: $x = 2\lambda^5$, $y = 20\lambda^2$, $z = \lambda^3$, $w = 2$.

It is thus seen that the tangents to the conical curve go into the points of the complex curve, and the osculating planes of the conical curve go into the tangents of the complex curve. The vertex goes into l. The points of c_{2a+b} go into lines of ψ cutting l, determined by the osculating plane. These lines form a ruled surface contained in a special linear congruence having l for single directrix, and the complex curve for asymptotic line. The order of the surface is $2a + b$; l is an a-fold directrix and an a-fold generator; b generators lie in any plane through l. The order of the complex curve is (generally) equal to the class of a plane section of this surface.

5. *Asymptotic lines on certain ruled surfaces.*

The asymptotic lines of an algebraic ruled surface contained in a linear congruence are algebraic, belong to a complex containing the congruence, cut every generator twice and are of order equal to the class of a plane section of the surface. We can therefore determine their genus by Segre's theorem.† If the surface have an m-fold directrix and a distinct n-fold directrix, δ double, and $\varkappa$ cuspidal generators, the genus p is defined by

$$p = 4mn - 3(m + n) + 1 - 4\delta - 5\varkappa.$$

The pinch-points are inflexions for all the asymptotic lines, the common inflexional tangent being the torsal generators. All the generators of the surface are principal chords, i. e., the osculating plane at each point of intersection passes through the other point.

If these curves be projected from any point on the m-fold directrix, the plane projection will have a $2(m^2 - m - \delta - \varkappa)$-fold point, each branch having

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* These two complex curves have other remarkable properties which were discussed in the *Journal*, vol. 28.

† *Math. Ann.* vol. 34, p. 3 (1889).

an inflexion at the multiple point and n other nodes lying on a line, the nodal tangents being inflexional. All the remaining points in which the line joining the nodes cuts the curves are points of inflexion. The various asymptotic lines of the surface will have the inflexional tangents at the multiple point in common, as well as the simple inflexions and the nodes, but the inflexional tangents at the nodes will form a quadratic involution.

Conversely, given any complex curve, we can construct a ruled surface having the given curve for asymptotic line. Let h be a line not belonging to the complex. Connect each point P of c with the point Q on l in which l pierces the osculating plane of P. The line PQ will generate a surface having h, h' for directrices, h' being the conjugate polar of h as to the complex. By choosing h in various ways, the given curve can be made of different order than that of the other asymptotic lines. In particular, the line can be so chosen that all the asymptotic lines are composite.

6. *Maximum genus of complex curves.*

The question of the maximum genus of a complex curve of given order is equivalent to that of the maximum number of cusps which a conical curve of given order can have. The latter in turn is replaced by the problem of plane curves: given a curve of order $2a$, having a branches touching each other at a given point, required the maximum number of cusps which the curve may have, since it is easy to see that we need only consider curves of complete intersection, which do not pass through the vertex of the cone. This problem has not been solved, it is simply a special case of the classic problem of the number of constants which the double points of a curve absorb, but an upper limit can be fixed in each case. Thus, if $a = 4$, $\beta \ngtr 7$; if $\beta = 7$, it can have no double points. This gives rise to a complex curve of order 7 and genus 2. If $a = 6$, $\beta = 18$, $H = 2$ exists and gives rise to a complex curve of order 8 and genus 5. It may be obtained as the intersection of K_2 and a cone having for base a curve of order 6 and class 3, two tangents lying in tangent planes of K_2. The asymptotic lines of the quartic scroll having two distinct double directrices are of this type. They have forty inflexional tangents, eight of which are generators of the scroll. In general, l is a bisecant and is intersected by 18 tangents, making its rank 24.

By reducing to simple curves of maximum genus we obtain the following upper limits, but for larger values of a the actual upper limit is probably lower:

a	2	3	4	5	6	7	8	9	10	11	12
β	1	4	7	12	18	26	36	47	59	72	88

The corresponding complex curves become:

order	3	4	5	6	7	8	9	10
max. genus	0	0	0	1	2	5	10	15

7. *Complex curves of orders three to six.*

Every twisted cubic belongs to a linear complex. The quartic with two inflexions is the only form, although the cuspidal quartic is selfdual in the more general sense. The complex quintics are unicursal; they have four inflexions, with or without a double tangent, a four-point tangent with two inflexions or another four-point tangent, a cusp with osculating plane having five-point contact and two inflexions, or another cusp. The forms having four inflexions, or one cusp can not lie on a quadric.

The conical curves which transform into rational sextics contained in ψ are of the forms:

(2, 1), $H = 2$, this goes into $v = 6$, l is $t + s_2$.
(2, 2), $\beta = 2$, $H = 1$, l is t_2.
(1, 4), all tangents coincident, l is tangent of 5-point contact.
(3, 0), $\beta = 1$, $H = 3$, l is s_3.
(3, 1), $\beta = 3$, $H = 3$, l is $t + s_1$.
(2, 3), $\beta = 2$, $H = 0$ (coincident tangents), l has 4-point coincident contact.
(1, 3), two coincident tangents, l is simple and distinct inflexional tangent.*

Of these forms, all but the first and last may also have a double osculating plane, giving rise to a bitangent of the complex curve. Stationary planes having 5-point contact can occur in all, and having 6-point contact in all but the first

* The statement made by Van der Vries, *Proceedings Amer. Acad.* vol. 38 (1903), p. 524, that no two of the tangents at a triple-point on a twisted quintic can coincide is incorrect. The equations of our curve may be written $x : y : z : w = \lambda^4 (\lambda - 1) : 2\lambda^3 (\lambda - 2)^2 : \lambda (\lambda - 1)^3 : 1$ in which case the complex curve has another tangent of four-point contact.

and last. Combining these with possible point singularities, we obtain the following forms:

$\alpha = \beta = 0$.

$v = 6$, $\omega = 0$ or 1.

$v = 4$, one 4-point contact tangent, $\omega = 0$ or 1, or one v is tangent.

$v = 2$, two 4-point contact tangents.

$v = 0$, three 4-point contact tangents.

$v = 3$, one 5-point contact tangent.

$v = 1$, one 4-, one 5-point contact tangent.

$v = 0$, two 5-point contact tangents.

$\alpha = \beta = 1$.

$v = 4$, $\omega = 0$ or 1.

$v = 2$, one 4-point.

$v = 0$, two 4-point.

$\alpha = \beta = 2$. $v = 2$; $v = 0$, one 4-point.

$\alpha = \beta = 3$. $v = 0$.

The sextics of genus 1 have already been discussed.

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